Lecture 2: Decomposition Matrices and Projective Indecomposable Characters

William Chuang, Based on Prof. Lux' Lectures and Navarro's Book

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1. Overview and Motivation

In modular representation theory, irreducible complex characters $\chi \in \mathrm{Irr}(G)$ reduce modulo p to (possibly reducible) Brauer characters $\phi \in \mathrm{IBr}(G)$. The relationship between these characters is encoded in the *decomposition matrix*, a central object in the theory.

2. Decomposition Matrix

Definition 1 (Decomposition Matrix)**.** *Let G be a finite group and p a prime dividing $|G|$. The* decomposition matrix $D = (d_{\chi\phi})$ *records the multiplicities with which each Brauer character $\phi \in \mathrm{IBr}(G)$ appears in the reduction modulo p of each ordinary irreducible character $\chi \in \mathrm{Irr}(G)$:*

$$\chi^\circ = \sum_{\phi \in \mathrm{IBr}(G)} d_{\chi\phi} \cdot \phi.$$

Here, χ° denotes the reduction of χ modulo p, and the coefficients $d_{\chi\phi}$ are non-negative integers.

3. The Space of Class Functions Vanishing on p-Singular Elements

Definition 2. *Let $\mathrm{VCF}(G)$ be the $\mathbb{C}$-vector space of class functions $f : G \to \mathbb{C}$ such that $f(x) = 0$ whenever $x \in G$ is p-singular (i.e., the order of x is divisible by p).*

The dimension of $\mathrm{VCF}(G)$ equals the number of p-regular conjugacy classes of G.

Theorem 1 (Navarro, Theorem 2.13)**.** *The set $\mathrm{IPr}(G)$ of projective indecomposable characters forms a $\mathbb{Z}$-basis for $\mathrm{VCF}(G)$.*

4. Inner Product on p-Regular Classes

Definition 3. *The inner product $\langle f, g \rangle_{p'}$ for class functions f, g on G is defined by:*

$$\langle f, g \rangle_{p'} = \frac{1}{|G|} \sum_{\substack{x \in G \\ x\ p\text{-regular}}} f(x)\overline{g(x)}|C_G(x)|.$$

This inner product restricts the summation to p-regular elements only.

Theorem 2. *With respect to* $\langle -, - \rangle_{p'}$*, the sets* $\mathrm{IBr}(G)$ *and* $\mathrm{IPr}(G)$ *form dual bases:*

$$\langle \phi, \Phi \rangle_{p'} = \delta_{\phi,\Phi}.$$

This implies any $f \in \mathrm{VCF}(G)$ can be expressed as:

$$f = \sum_{\phi \in \mathrm{IBr}(G)} \langle f, \phi \rangle_{p'} \cdot \Phi_\phi,$$

where $\Phi_\phi \in \mathrm{IPr}(G)$ is the projective indecomposable character associated with ϕ.

5. Matrix Formulation

Let:

- Φ: the character table of $\mathrm{IPr}(G)$,
- Ψ: the character table of $\mathrm{IBr}(G)$,
- C: the change-of-basis matrix from $\mathrm{IBr}(G)$ to $\mathrm{Irr}(G)$,
- M: the decomposition matrix $(d_{\chi\phi})$.

Then:

$$\Phi = C \cdot \Psi, \quad \text{or equivalently,} \quad C = \Phi \cdot \Psi^{-1}.$$

6. Dickson's Lemma (Navarro Lemma 2.14)

Lemma 1 (Dickson's Lemma)**.** *Let* $\phi \in \mathrm{IBr}(G)$ *and let* $P \leq G$ *be a Sylow* p*-subgroup. Then the restriction of* ϕ *to* P *satisfies:*

$$\phi|_P = \frac{\phi(1)}{|P|} \sum_{x \in P} \tau(x),$$

where τ *is a generalized character of* P*.*

This lemma expresses the behavior of Brauer characters upon restriction to p-subgroups.

7. Summary

- The decomposition matrix connects ordinary and Brauer characters.
- The space $\mathrm{VCF}(G)$ has a basis given by projective indecomposable characters.
- The inner product on p-regular classes allows us to interpret $\mathrm{IBr}(G)$ and $\mathrm{IPr}(G)$ as dual bases.
- Dickson's Lemma gives a description of how Brauer characters restrict to p-subgroups.

Next Lecture (Lecture 3 Preview)

In Lecture 3, we will:

- Investigate the structure and properties of decomposition matrices,
- Introduce block theory and defect groups,
- Analyze examples, especially from symmetric groups and abelian defect blocks.